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# THE EFFECT OF MODAL COUPLING IN RANDOM VIBRATION ANALYSIS

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When modal analysis techniques are used to determine spectral densities of the responses of distributed-parameter linear mechanical structures subjected to stationary random excitation, a summation of all modal spectral densities (both direct- and cross-) should be performed. A very common textbook recommendation is that the modal cross-spectral densities may be neglected if certain conditions are satisfied. In this paper that recommendation will be discussed. Using a simple example the influence of modal cross-spectral densities on the spectral densities of some responses of a simply supported beam will be investigated. Response standard deviations will be determined and the importance of the modal cross-spectral densities in some frequency ranges, covering also resonance frequencies of the beam, will be demonstrated. Special interest is devoted to extreme values of some response processes. 

## 1. INTRODUCTION

When solving stationary random vibration problems by the use of modal analysis, direct- and cross-spectral densities of the modal co-ordinates will be obtained. To find the spectral density of a structural response in the physical co-ordinates, all the modal spectral densities (direct- and cross-) should be summed. In the literature, it is sometimes stated that the modal cross-spectral densities may be neglected if

(a) the system is lightly damped, and
(b) the eigenfrequencies of the system are well separated,

see, for example, references [1–6].

The first one to point out conditions under which the modal cross-spectral densities may be neglected seems to have been Bolotin, who already in 1961 stated that the conditions for possible neglection of the modal cross-terms are violated in two cases, namely when the damping is very large or when the elastic system has multiple or closely spaced eigenfrequencies. The conditions to be satisfied read [7]:

$$\max(\omega_j \zeta_j, \omega_k \zeta_k) \ll |\omega_j^2 - \omega_k^2|,$$

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where $\zeta_i$ is relative damping in mode $i$, and $\omega_i$ is the $i$th eigenfrequency of the structure. This statement seems to have been interpreted so that the cross-terms may always be neglected if the conditions are satisfied. As will be shown below this is not the case. The conditions might be necessary, but not sufficient. One indication of this problem is given in reference [5], where the modal closeness is commented upon: "No general operational definition of closeness exists".

The role of the cross-spectral densities has been discussed by several authors, see for example references [7–11]. In reference [9], Elishakoff uses a single mass constrained by two springs in orthogonal directions for illustrating the significance of cross-correlations in random vibration analysis. Crandall [10] noticed that localization of response occurred in taut strings, beams, membranes and plates subjected to broad-band random vibration. The localization of response occurred at the driving point and at its symmetric counterparts of the structure. In a shell, the localization effect may be hardly detected if the modal cross-terms are omitted, see reference [11], where a survey and discussion of the importance of the modal cross-terms is given.

In reference [12], examples were given where modal cross-spectral densities were shown to play an important role in the response spectral densities of a structure, although the Bolotin conditions were satisfied. These examples concerned structural responses to narrow-band random excitation. The structural parameters were such that an anti-resonance of the structure appeared at, or in the vicinity of, the frequency range of the random excitation. In such a case, none of the modes of the structure dominates in the response, and consequently, the contributions from the cross-spectral densities may be important. It was demonstrated that modal cross-spectral densities contribute to the response of a structure mainly in the frequency range close to anti-resonances of the structure. This has also been pointed out in, for example reference [13], where the effects of modal coupling on the acoustic power radiation from panels were investigated. In reference [13] it was found that under off-resonant excitation the contributions due to modal coupling may be important.

In reference [14], some other examples were given where modal cross-spectral densities played an important role in the structural response although the conditions of small damping and well separated eigenfrequencies were satisfied. The bandwidth of the excitation was broadband, so that it covered several resonance frequencies of the structure (a taut string). The excitation was distributed along the whole structure and not, as in Crandall's problem, concentrated to one single point. In reference [14] a band-limited white noise excitation, covering the eigenfrequencies eight to 12 of the string, was used. It was demonstrated that the approximate mean square value of the string displacement, obtained by neglecting the modal cross-spectral densities, could differ from the exact mean square value by a factor of three (or even more).

A similar problem of decoupling of modes arises in structures that are non-proportionally damped, see references [15–18]. The simplest and the most common approach to this problem is to neglect the off-diagonal terms and replace the full damping matrix with a diagonal one. The two conditions given above, (a) and (b), should then be satisfied [16]. As pointed out in references [15, 18], the input frequency may have a significant effect on the approximation error obtained when neglecting the off-diagonal terms, c.f., reference [12]. In references [17, 18], it was also pointed out that even if the approximation error is small in the modal co-ordinates, the error in the physical co-ordinates need not be small.

In many applications, the distribution of extreme values (maxima and minima) of a structural response is important. For example, a small error of the extreme values of the stresses in a structure may influence the fatigue life of the structure considerably. When calculating extreme values of a random process, the bandwidth of the process will play an important role. It will be demonstrated that neglect of the modal cross-spectral terms may result in an error of the process bandwidth that is, in the example studied here, of the order of $-20$ to $+55$ per cent.

In the following sections, the response of a simply supported beam subjected to a stationary random loading will be investigated by use of the modal analysis technique. Spectral densities are plotted and standard deviations are calculated. Results where the modal cross-spectral densities have been neglected are compared with corresponding exact results ("exact" in the meaning that the modal cross-spectral densities are included—but still approximate due to mode truncation). As will be demonstrated, the difference between an approximate response and the corresponding exact one may be quite large at some positions along the beam.

## 2. ANALYSIS

### 2.1. RANDOM EXCITATION

Consider the random vibration of a simply supported beam of length $L$. The excitation of the beam is a stationary random transverse force $f(x, t)$ per unit length. In a general case, the force could be random along the length of the beam and random in time.

First, a force $f(t)$ ($t$ is time) is considered. The autocorrelation function of the force $f(t)$ will be denoted by $R_{ff}(\tau)$ where $\tau$ is time separation. One has $R_{ff}(\tau) = E[f(t)\, f(t+\tau)]$ where $E$ stands for expectation.

The one-dimensional spectral density of the force $f(t)$ will be denoted by $S_{ff}(\omega)$ where $\omega$ is time frequency in radians per second. One has

$$S_{ff}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{ff}(\tau)\,\mathrm{e}^{-\mathrm{i}\omega\tau}\,\mathrm{d}\tau. \tag{1}$$

The auto-correlation function $R_{ff}(\tau)$ is

$$R_{ff}(\tau) = \int_{-\infty}^{\infty} S_{ff}(\omega)\,\mathrm{e}^{\mathrm{i}\omega\tau}\,\mathrm{d}\omega. \tag{2}$$

Relationships (1) and (2) thus form a one-dimensional Fourier transform pair.

Similarly, for a process which is random in space (a random function $f(x)$ of distance $x$, say) one obtains the autocorrelation function $R_{ff}(\xi) =$

$E[f(x)f(x+\xi)]$ where $\xi$ is spatial separation. The one-dimensional spectral density of the function $f(x)$ will be denoted by $S_{ff}(\gamma)$, where $\gamma$ is spatial frequency in radians per meter ($\gamma$ is also called wave *number*, the wave*length* $\lambda$ is $\lambda = 2\pi/\gamma$). The spectral density $S_{ff}(\gamma)$ is defined as in equation (1) ($\gamma$ replaces $\omega$ and $\xi$ replaces $\tau$).

The two-dimensional correlation function of the force $f(x, t)$ will be denoted by $R_{ff}(\xi, \tau)$ where $\xi$ is spatial separation and $\tau$ is time separation. One has

$$R_{ff}(\xi, \tau) = E[f(x, t)f(x+\xi, t+\tau)]. \tag{3}$$

The two-dimensional spectral density of the force $f(x, t)$ will be denoted by $S_{ff}(\gamma, \omega)$, where $\gamma$ and $\omega$ are defined above. By definition one has

$$S_{ff}(\gamma, \omega) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{ff}(\xi, \tau)\,\mathrm{e}^{-\mathrm{i}(\gamma\xi+\omega\tau)}\,\mathrm{d}\xi\,\mathrm{d}\tau. \tag{4}$$

The cross-correlation function $R_{ff}(\xi, \tau)$ can, also by definition, be written

$$R_{ff}(\xi, \tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_{ff}(\gamma, \omega)\,\mathrm{e}^{\mathrm{i}(\gamma\xi+\omega\tau)}\,\mathrm{d}\gamma\,\mathrm{d}\omega. \tag{5}$$

Relationships (4) and (5) thus form a two-dimensional Fourier transform pair.

The study presented here will be restricted to a force that is constant along the length $L$ of the beam and stationary random in time $t$, so that $f(x, t) = f(t)$, which inserted into equation (3) gives $R_{ff}(\xi, \tau) = E[f(t)f(t+\tau)] = R_{ff}(\tau)$. The two-dimensional spectral density $S_{ff}(\gamma, \omega)$ of the force $f(x\,t) = f(t)$ is then obtained, by equation (4), as

$$S_{ff}(\gamma, \omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{ff}(\tau)\,\mathrm{e}^{-\mathrm{i}\omega\tau}\,\mathrm{d}\tau \times \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\gamma\xi}\,\mathrm{d}\xi = S_{ff}(\omega)\delta(\gamma), \tag{6}$$

where $\delta(\,)$ is the Dirac delta function.

By use of $R_{ff}(\xi, \tau) = R_{ff}(\tau)$ and the relationships equations (2) and (5) it can be seen that, for $f(x, t) = f(t)$,

$$S_{ff}(\omega) = \int_{-\infty}^{\infty} S_{ff}(\gamma, \omega)\,\mathrm{e}^{\mathrm{i}\gamma\xi}\,\mathrm{d}\gamma. \tag{7}$$

By introducing the spatial separation $\xi = s_2 - s_1$ into equation (7), one obtains the cross-spectral density between values of the random function $f(x, t)$ at the two points $x = s_1$ and $s_2$. The cross-spectral density, here called $S_{f_{s1}f_{s2}}(s_1, s_2, \omega)$, becomes, by the use of equation (6),

$$S_{f_{s1}f_{s2}}(s_1, s_2, \omega) = \int_{-\infty}^{\infty} S_{ff}(\omega)\,\delta(\gamma)\,\mathrm{e}^{\mathrm{i}\gamma(s_2-s_1)}\,\mathrm{d}\gamma = S_{ff}(\omega)\mathrm{e}^{0}. \tag{8}$$

Thus, when the excitation $f(x, t)$ is $f(x, t) = f(t)$, one obtains $S_{f_{s1}f_{s2}}(s_1, s_2, \omega) = S_{ff}(\omega)$.

Finally, if the excitation is random white noise (uncorrelated in time, band-limited or not), then the spectral density $S_{ff}(\omega)$ is a constant, $S_0$ say, so that $S_{ff}(\omega) = S_0$.

Next, turn to the response of the beam due to white noise excitation.

### 2.2. BEAM RESPONSE

Let $y(x, t)$ be the random response of a simply supported beam subjected to the excitation $f(x,t)$. The cross-spectral density $S_{y_{x1}y_{x2}}(x_1, x_2, \omega) = S_{yy}(x_1, x_2, \omega)$ (index on $y$ will be omitted from now on) between values of the response function $y(x, t)$ at the two points $x = x_1$ and $x_2$ will be determined.

Let $H(x_i, s_j, \omega)$ be the frequency response function giving the response $y(x, t)$ at point $x = x_i$ due to a unit harmonic input force $F(t)$ at point $x = s_j$. In the case of $N$ separate input forces $F_{sj}(s_j, t)$, $j = 1, \ldots, N$, the cross-spectral density $S_{yy}(x_1, x_2, \omega)$ between values of the response function $y(x, t)$ at the two points $x_1$ and $x_2$ becomes (* indicates complex conjugate)

$$S_{yy}(x_1, x_2, \omega) = \sum_{j=1}^{N} \sum_{k=1}^{N} H^*(x_1, s_j, \omega) H(x_2, s_k, \omega)\, S_{F_{sj}F_{sk}}(s_j, s_k, \omega). \tag{9}$$

Let one input force $F_{sj}(s_j, t)$ (in Newtons) represent the excitation from the distributed load $f(x, t)$ (in Newtons per meter) at position $x = s_j$ along the length $\Delta s_j$ of the beam. One then obtains the approximation $F_{sj}(s_j, t) = f(s_j, t)\Delta s_j$. The relationship between the cross-spectral density of the point forces $F_{sj}$ at $s_j$ and $F_{sk}$ at $s_k$ and the cross-spectral density of the distributed load $f(s, t)$ at the same points is $S_{F_{sj}F_{sk}}(s_j, s_k, \omega) = S_{f_{sj}f_{sk}}(s_j, s_k, \omega)\Delta s_j \Delta s_k$ [4], which together with equation (9) gives

$$S_{yy}(x_1, x_2, \omega) = \sum_{j=1}^{N} \sum_{k=1}^{N} H^*(x_1, s_j, \omega) H(x_2, s_k, \omega)\, S_{f_{sj}f_{sk}}(s_j, s_k, \omega)\Delta s_j \Delta s_k. \tag{10}$$

For $N \to \infty$ (implying $\Delta s_j \to 0$ and $\Delta s_k \to 0$) the relationship (10) becomes

$$S_{yy}(x_1, x_2, \omega) = \int_0^L \int_0^L H^*(x_1, s_j, \omega) H(x_2, s_k, \omega)\, S_{f_{sj}f_{sk}}(s_j, s_k, \omega)\, \mathrm{d}s_j \mathrm{d}s_k. \tag{11}$$

Now introduce the excitation $f(x, t) = f(t)$, which is also uncorrelated in time so that $S_{ff}(\omega) = S_0$. Then, according to equation (8), $S_{f_{sj}f_{sk}}(s_j, s_k, \omega) = S_{ff}(\omega) = S_0$, and one obtains the cross-spectral density

$$S_{yy}(x_1, x_2, \omega) = S_0 \int_0^L H^*(x_1, s_j, \omega)\, \mathrm{d}s_j \int_0^L H(x_2, s_k, \omega)\, \mathrm{d}s_k. \tag{12}$$

From equation (12), a direct spectral density $S_{yy}(x, \omega)$ is obtained by simply substituting $x_1 = x_2 = x$.

### 2.3. FREQUENCY RESPONSE FUNCTION $H$

The relationship (12) will be further investigated below, but first the frequency response function $H(x_i, s_j, \omega)$ will be determined by the use of modal analysis. The frequency response function $H(x_i, s_j, \omega)$ gives the response $y(x, t)$ at $x = x_i$ to a unit harmonic input force $F(t) = F_0 e^{i\omega t}$ at $x = s_j$ ($F_0 = 1$ N). Thus, $y(x, t) = H(x, s_j, \omega) \times F_0 e^{i\omega t}$ for any $x \in [0, L]$.

The equation of motion of a beam (bending stiffness $EI$, damping $c$, mass $m$ per unit length, and length $L$) loaded by a transverse force $f(x, t)$ is

$$EI \frac{\partial^4 y}{\partial x^4} + c \frac{\partial y}{\partial t} + m \frac{\partial^2 y}{\partial t^2} = f(x, t). \tag{13}$$

The angular eigenfrequencies $\omega e_l$ of an undamped simply supported beam are

$$\omega_{el} = l^2 \pi^2 \sqrt{\frac{EI}{mL^4}}, \quad \text{where } l = 1, 2, 3, \ldots, \tag{14}$$

and the eigenmodes become

$$\Psi_l(x) = A_l \sin\left( l\pi \frac{x}{L} \right) \quad \text{for } l = 1, 2, 3, \ldots, \tag{15}$$

where the vibration amplitudes $A_l$ are to be determined.

To determine the frequency response function $H(x_i, s_j, \omega)$, the beam should be excited by a unit harmonic force with frequency $\omega$ at the position $x = s_j$. The excitation force $f(x, t)$ may then be written $f(x, t) = F_0 \delta(x - s_j) e^{i\omega t}$ where $\delta()$ is the Dirac delta function. The harmonic (frequency $\omega$) response $y(x, t)$ to this excitation may be written in terms of the undamped natural modes (eigenmodes) of the beam. One obtains

$$y(x, t) = \sum_{l=1}^{\infty} \Psi_l(x) e^{i\omega t} = \sum_{l=1}^{\infty} A_l \sin\left( l\pi \frac{x}{L} \right) e^{i\omega t}, \tag{16}$$

which in the beam equation (13) gives ($e^{i\omega t}$ omitted)

$$\sum_{l=1}^{\infty} \left\{ EI \left( l \frac{\pi}{L} \right)^4 + c\, i\omega - m\omega^2 \right\} A_l \sin\left( l\pi \frac{x}{L} \right) = F_0\, \delta(x - s_j). \tag{17}$$

Multiplying equation (17) by $\sin(k\pi x/L)$ and integrating over the length $L$ gives

$$\left\{ EI \left( l \frac{\pi}{L} \right)^4 + c\, i\omega - m\omega^2 \right\} A_l \frac{L}{2} = F_0 \sin\left( l \frac{\pi}{L} s_j \right), \tag{18}$$

from which $A_l$ is solved. One obtains

$$A_l = \frac{2 \sin(l\pi s_j / L)}{Lm\{\omega_{el}^2 + i\omega c/m - \omega^2\}} F_0. \tag{19}$$

The harmonic response $y(x, t)$ to the excitation force $F(t) = F_0 e^{i\omega t}$ at $x = s_j$ may now be summarized by the use of equations (16) and (19). One finds

$$y(x, t) = \sum_{l=1}^{\infty} \frac{2 \sin(l\pi s_j/L) \sin(l\pi x/L)}{Lm\{\omega_{el}^2 + i\omega c/m - \omega^2\}} F_0 e^{i\omega t}. \tag{20}$$

Thus, the frequency response function $H(x_i, s_j, \omega)$ is, with $c/m = \beta$,

$$H(x_i, s_j, \omega) = \sum_{l=1}^{\infty} \frac{2 \sin(l\pi s_j/L) \sin(l\pi x_i/L)}{Lm\{\omega_{el}^2 + i\omega\beta - \omega^2\}}. \tag{21}$$

## 2.4. RESPONSE SPECTRAL DENSITY $S$

The cross-spectral density $S_{yy}(x_1, x_2, \omega)$ of the response function $y(x,t)$ at the two points $x_1$ and $x_2$ can now be expressed in terms of the undamped natural modes (the eigenmodes) of the beam. Assume that the excitation $f(x, t) = f(t)$ is uncorrelated in time, so that $S_{ff}(\omega) = S_0$. Equations (12) and (21) then give

$$S_{yy}(x_1, x_2, \omega) = S_0 \times$$

$$\int_0^L \sum_{l=1}^{\infty} \frac{2 \sin(l\pi s_j/L) \sin(l\pi x_1/L)}{Lm\{\omega_{el}^2 - i\omega\beta - \omega^2\}} ds_j \int_0^L \sum_{n=1}^{\infty} \frac{2 \sin(n\pi s_k/L) \sin(n\pi x_2/L)}{Lm\{\omega_{en}^2 + i\omega\beta - \omega^2\}} ds_k. \tag{22}$$

Noting that

$$\int_0^L \sin\left(l\pi \frac{s_j}{L}\right) ds_j = \begin{cases} \dfrac{2L}{l\pi} & \text{if } l \text{ odd}, \\ 0 & \text{if } l \text{ even}, \end{cases} \tag{23}$$

one obtains

$$S_{yy}(x_1, x_2, \omega) = \frac{16 S_0}{\pi^2 m^2} \sum_{l=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} \frac{\sin(l\pi x_1/L) \sin(n\pi x_2/L)}{ln\{\omega_{el}^2 - i\omega\beta - \omega^2\}\{\omega_{en}^2 + i\omega\beta - \omega^2\}}. \tag{24}$$

This is the complete expression of the direct- and cross-spectral densities of the displacement response of a beam subjected to white noise loading $f(x, t) = f(t)$. A direct spectral density is obtained simply by putting $x_1 = x_2$. Only symmetric modes are included because of the symmetric loading of the beam. An approximation and simplification of this expression will now be discussed.

### 2.4.1. *Approximate spectral density*

The influence of the modal cross-spectral terms (the product terms $l \neq n$ in equation (24)) on the response spectral density $S_{yy}(x_1, x_2, \omega)$ will now be investigated. If the damping is small and if the spacing between adjacent eigenfrequencies is large, then the modal cross-spectral terms in equation (24) may (sometimes) be neglected, and one obtains

$$S_{yy}(x_1, x_2, \omega)_{\text{approx}} = \frac{16S_0}{\pi^2 m^2} \sum_{l=1,3,5,\ldots}^{\infty} \frac{\sin(l\pi x_1/L)\sin(l\pi x_2/L)}{l^2\{(\omega_{\text{el}}^2 - \omega^2)^2 + \beta^2\omega^2\}}. \tag{25}$$

The exact spectral density, equation (24), and its approximation, equation (25), will be investigated in connection with some numerical examples.

## 2.5. NUMERICAL EVALUATIONS

In all curves given below, the following numerical values have been used: bending stiffness $EI = 1$ (N m$^2$), length $L = 1$ (m), mass per unit length $m = \pi^4$ (kg/m). These numbers have been selected to give the undamped eigenfrequency $\omega_{\text{el}} = 1$ rad/s. The damping parameter $\beta = 0{\cdot}5$ (1/s) has been used throughout. If nothing else is stated, the number of modes used is 30, i.e., summation is made over the eigenfrequencies $l$ where $l = 1, 3, 5, \ldots, 59$.

The damping parameter $\beta$ may be expressed as a fraction of critical damping. For a one-degree-of-freedom system the critical damping is $c_{\text{crit}} = 2\sqrt{kM} = 2M\omega_{\text{e}}$, where $k$ is stiffness, $M$ is mass and $\omega_{\text{e}}$ is the undamped eigenfrequency $\omega_{\text{e}} = \sqrt{k/M}$. When the modal damping is critical, the value of the damping parameter $\beta = \beta_{\text{crit}}$ may be expressed as ($\beta$ is damping per meter and $m$ is mass per meter)

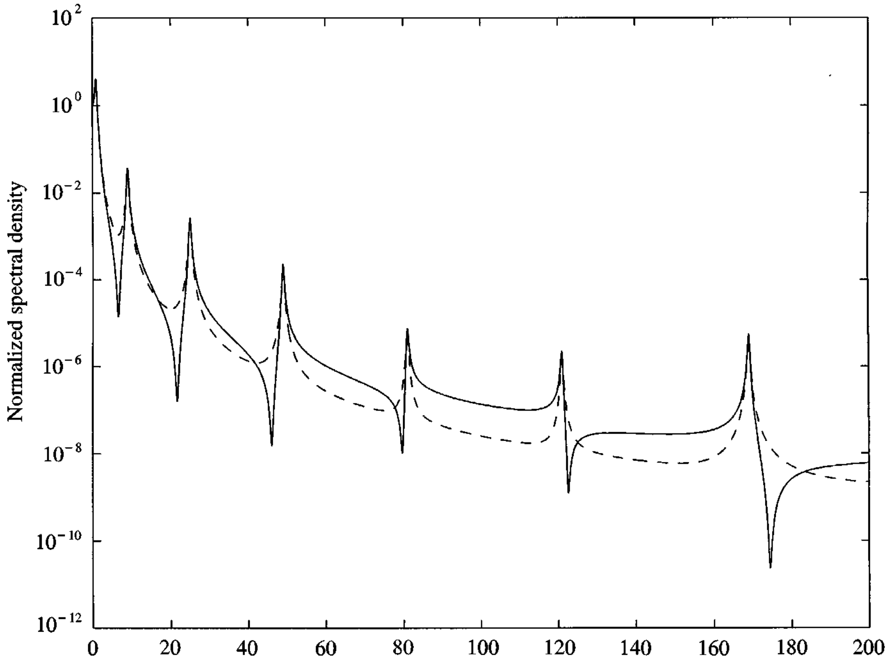
$$\beta_{\text{crit}} = \frac{c_{\text{crit}}}{m} = \frac{2m\omega_{el}}{m} = 2l^2\pi^2\sqrt{EI/mL^4}. \tag{26}$$

The damping $\beta = 0{\cdot}5$ thus gives 25 per cent of critical damping in mode one, 6·25 per cent of critical damping in mode 2 ($j = 2$), and so on.

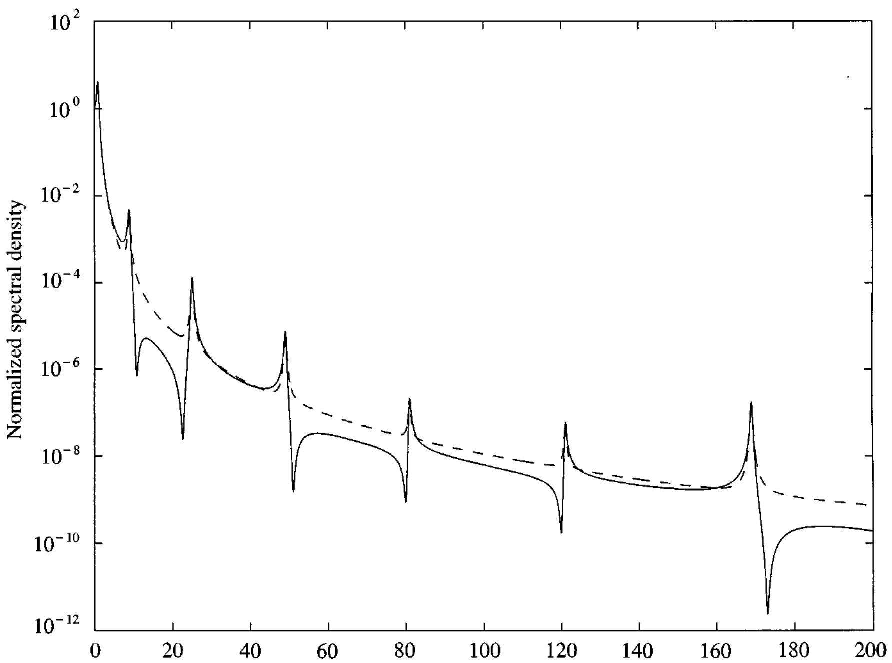
Direct ($x_1 = x_2 = x$) spectral densities $S_{yy}(x_1, x_2, \omega) = S_{yy}(x, \omega)$, normalized with respect to $S_{yy}(x, 0)$, have been plotted for $x = 0{\cdot}1L$ and $x = 0{\cdot}45L$ (Figures 1(a, b)) for the frequency range $\omega = 0$–200 rad/s. The full line curves in the figures include the modal cross-spectral terms as given by equation (24) and the dashed curves give the approximation, equation (25).

It is noted from Figure 1 that the modal cross-spectral terms are important if the spectral density has to be calculated between the resonance frequencies $\omega_{el}$. Close to the resonance frequencies the terms in the approximation, equation (25), are the most important, of course, and the cross-spectral terms (the double summation) in equation (24) may be neglected.

An undamped system has the so-called anti-resonance frequencies between the resonance frequencies. These anti-resonance frequencies cannot be detected when the approximation (25) is used, see Figure 1. It is also seen in the figure that for large frequency ranges ($\omega$ equals 50–170 rad/s, for example) the approximate solution lies almost entirely on one side of the exact solution (below the exact solution at $x = 0{\cdot}1L$ and above at $x = 0{\cdot}45L$). At the frequency $\omega = 100$ rad/s, for example, the




(a)




(b)

Figure 1. Normalized spectral densities $S_{yy}(x, \omega)/S_{yy}(x, 0)$ plotted for (a) $x = 0{\cdot}1L$ and (b) $x = 0{\cdot}45L$. The full line curves include modal cross-spectral densities as given in equation (24), and the dashed curves give the approximate solution according to equation (25).

ratio of the two solutions (exact/approximate) is 5·33 at $x = 0{\cdot}1L$ and 0·56 at $x = 0{\cdot}45L$.

### 2.5.1. *Standard deviation of response*

The influence of the modal cross-spectral terms on the standard deviation $\sigma_y(x)$ of the deflection $y(x,t)$ will now be investigated. The standard deviation of the response is obtained from the mean square value of $y(x,t)$ as $\sigma_y(x) = (E[y^2(x,t)])^{1/2}$ (the mean value of the process is zero here, giving that the standard deviation equals the r.m.s. value). The mean square value is obtained by the integration of the spectral density $S_{yy}(x,\omega)$ over the frequency $\omega$. The direct spectral density $S_{yy}(x,\omega)$ is obtained from equations (24) and (25), by putting $x_1 = x_2 = x$. One has

$$E[y^2(x,t)] = \int_{-\infty}^{\infty} S_{yy}(x,\omega)\,\mathrm{d}\omega. \tag{27}$$

Now suppose that the excitation of the beam is band-limited white noise such that

$$S_{ff}(\omega) = \begin{cases} S_0 & \text{if } \omega_1 \leqslant |\omega| \leqslant \omega_2, \\ 0 & \text{elsewhere.} \end{cases} \tag{28}$$

This implies that the integral in equation (27) should be evaluated between $-\omega_2$ and $-\omega_1$ and between $\omega_1$ and $\omega_2$. The frequency interval of excitation selected is from $\omega_1 = 50$ rad/s to $\omega_2 = 160$ rad/s. This frequency range covers the eigenfrequencies eight to 12 of the beam (of which only nine and 11 are excited here).

The integration limits yield a bandwidth $\varepsilon$ ($\varepsilon$ defined below in equation (33)) of the excitation process as $\varepsilon = 0{\cdot}49$, which is neither extremely narrow-band ($\varepsilon = 0$), nor extremely broadband ($\varepsilon = 1$).

The imaginary part of a (cross-)spectral density is an odd function of $\omega$. When integrated over an even interval the imaginary part of the integral becomes zero. The real part of the spectral density is an even function of $\omega$. The mean square value in equation (27) may then be calculated as ($\Re$ means "real part of")

$$E[y^2(x,t)]_{\text{exact}} = \frac{32S_0}{\pi^2 m^2} \times$$

$$\int_{\omega_1}^{\omega_2} \Re \sum_{l=1,3,\dots}^{\infty} \sum_{n=1,3,\dots}^{\infty} \frac{\sin(l\pi x/L)\sin(n\pi x/L)\,\mathrm{d}\omega}{ln\{\omega_{el}^2 - \mathrm{i}\omega\beta - \omega^2\}\{\omega_{en}^2 + \mathrm{i}\omega\beta - \omega^2\}}. \tag{29}$$

When the modal cross-spectral terms in equation (29) are neglected, the expression simplifies to

$$E[y^2(x,t)]_{\text{approx}} = \frac{32S_0}{\pi^2 m^2} \int_{\omega_1}^{\omega_2} \sum_{l=1,3,\dots}^{\infty} \frac{\sin^2(l\pi x/L)\,\mathrm{d}\omega}{l^2\{(\omega_{el}^2 - \omega^2)^2 + \omega^2\beta^2\}}. \tag{30}$$

The square root of the two expressions, equations (29) and (30), will now be compared. The mean square value $E[y^2(x,t)]$ of the response $y(x,t)$ has been calculated as a function of $x$ for $0 < x \leqslant L/2$. Due to symmetry one has $E[y^2(x,t)] = E[y^2(L-x,t)]$. The mean square value has been normalized with respect to the factor $32S_0/\pi^2m^2$ (thus, not the same as in Figure 1). The root of the normalized mean square value gives the (normalized) standard deviation $\sigma_y$, which is then plotted.

In Figure 2(a) the (normalized) standard deviation $\sigma_y$ of the beam deflection is plotted as a function of position $x/L$ along the beam (note the symmetry for $x/L > 0{\cdot}5$). In Figure 2(b) the percentage error, $100(\sigma_y^{approx} - \sigma_y^{exact})/\sigma_y^{exact}$, has been plotted. It is seen that this error may be quite large: approximately $-40$ per cent at $x = 0{\cdot}1L$ and $+60$ per cent at $x = 0{\cdot}45L$.

In acoustic radiation, the deflection velocity $\dot{y}(x,t)$ plays an important role [13]. The mean square spectral density of $\dot{y}(x,t)$ is obtained from the spectral density of the deflection $y(x,t)$ as

$$S_{\dot{y}\dot{y}}(x,\omega) = \omega^2 S_{yy}(x,\omega). \tag{31}$$

Integration of equation (31) over the same frequency range as above yields exact and approximate standard deviations $\sigma_{\dot{y}}$ as given in Figure 3a (normalized as $\sigma_y$). The relative error of the approximate solution is given in Figure 3b. Also here, the relative error is quite large at certain positions along the beam.

For the beam acceleration $\ddot{y}(x,t)$, the result is similar: the approximate value of the standard deviation $\sigma_{\ddot{y}}$ may deviate as much as $\pm 30$ per cent from the exact value.

## 2.6. EXTREME VALUES

In many practical situations, the extreme values of a random process may be of great interest (for example in ocean and civil engineering, fatigue calculations, and so on). In the last part of this study, the extreme values of the beam deflection will be investigated. The expected mean value of the maximum beam deflection during a time period $T$ may be calculated as a factor (the peak factor) multiplying the standard deviation $\sigma_y$. One has ($\gamma$ is the Euler constant, $\gamma = 0{\cdot}57721\ldots$)

$$E[y_{max}(x)] = \left(\sqrt{2\ln N_{mean}} + \frac{\gamma}{\sqrt{2\ln N_{mean}}} + \ldots\right)\sigma_y. \tag{32}$$

The time period $T$ should be large, so that $\ln N_{mean} \gg 1$ (if not, another formula should replace equation (32), [19]). $N_{mean}$ is the number of mean value (here zero) up-crossings of the process during the time period $T$. One has

$$N_{mean} = N_{max}\sqrt{1-\varepsilon^2} = \frac{1}{2\pi}\frac{\sigma_{\ddot{y}}}{\sigma_{\dot{y}}}T\sqrt{1-\varepsilon^2}, \tag{33a}$$

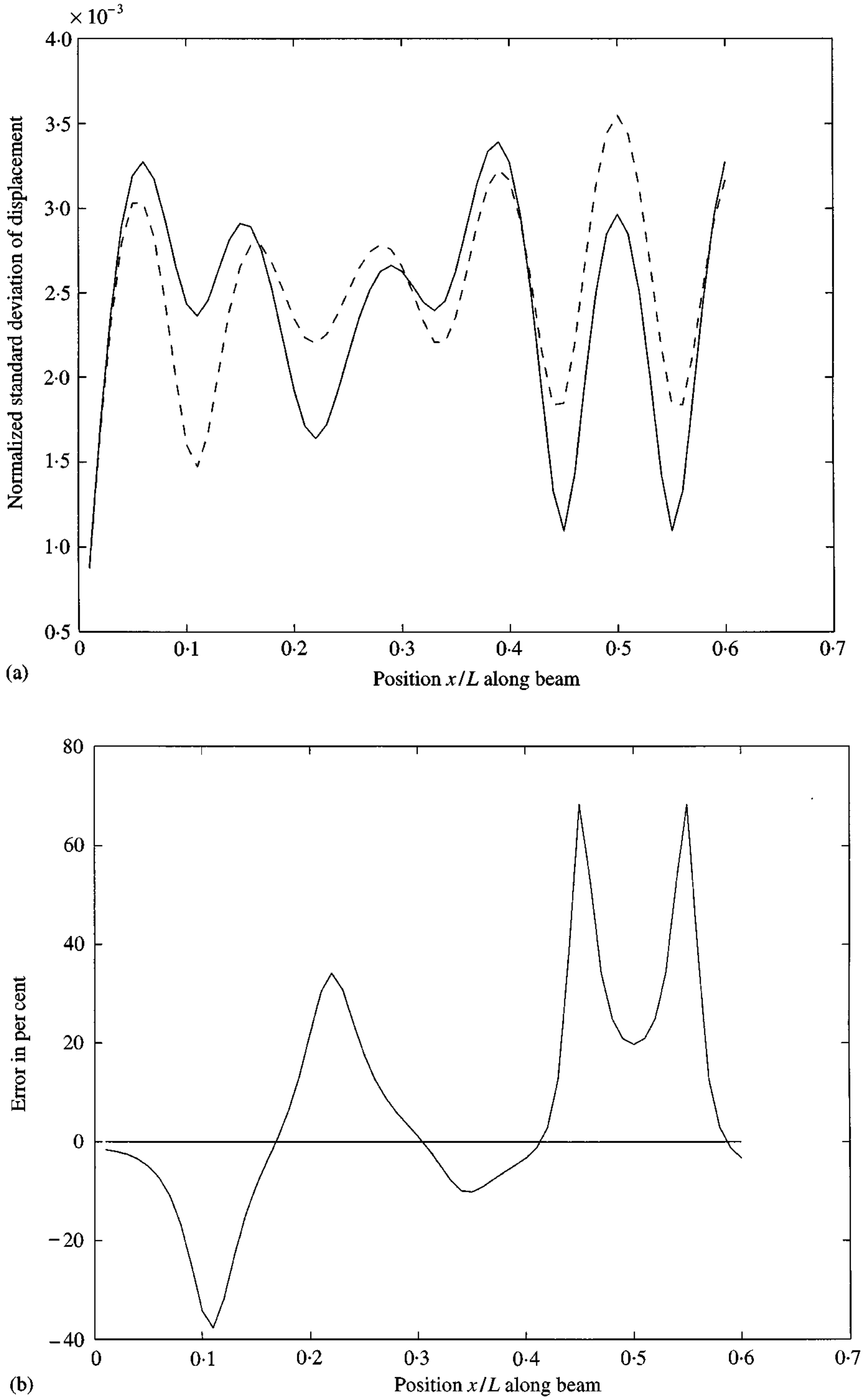



Figure 2. (a) Normalized standard deviation $\sigma_y$ of beam deflection $y(x,t)$ for $x/L < \frac{1}{2}$ (symmetry for $x/L > \frac{1}{2}$). The full line curve includes the modal cross-spectral terms as given in equation (24), and the dashed curve gives the approximate solution according to equation (25). (b) Error (in per cent) obtained when modal cross-spectral terms are neglected.

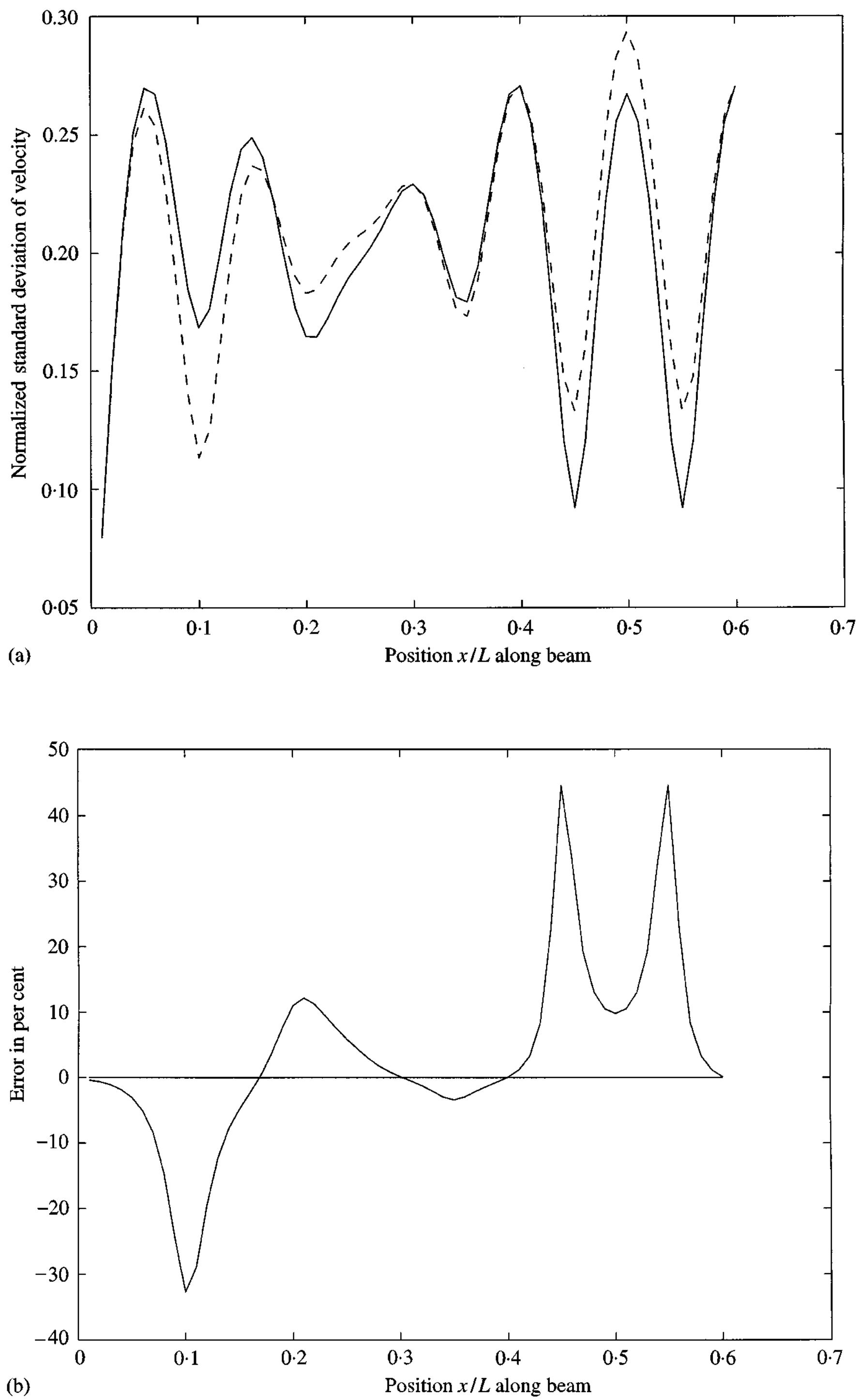



Figure 3. (a) Normalized standard deviation $\sigma_{\dot{y}}$ of beam deflection $\dot{y}(x, t)$ for $x/L < \frac{1}{2}$ (symmetry for $x/L > \frac{1}{2}$). The full line curve includes the modal cross-spectral terms as given in equation (24), and the dashed curve gives the approximate solution according to equation (25). (b) Error (in per cent) obtained when modal cross-spectral terms are neglected.

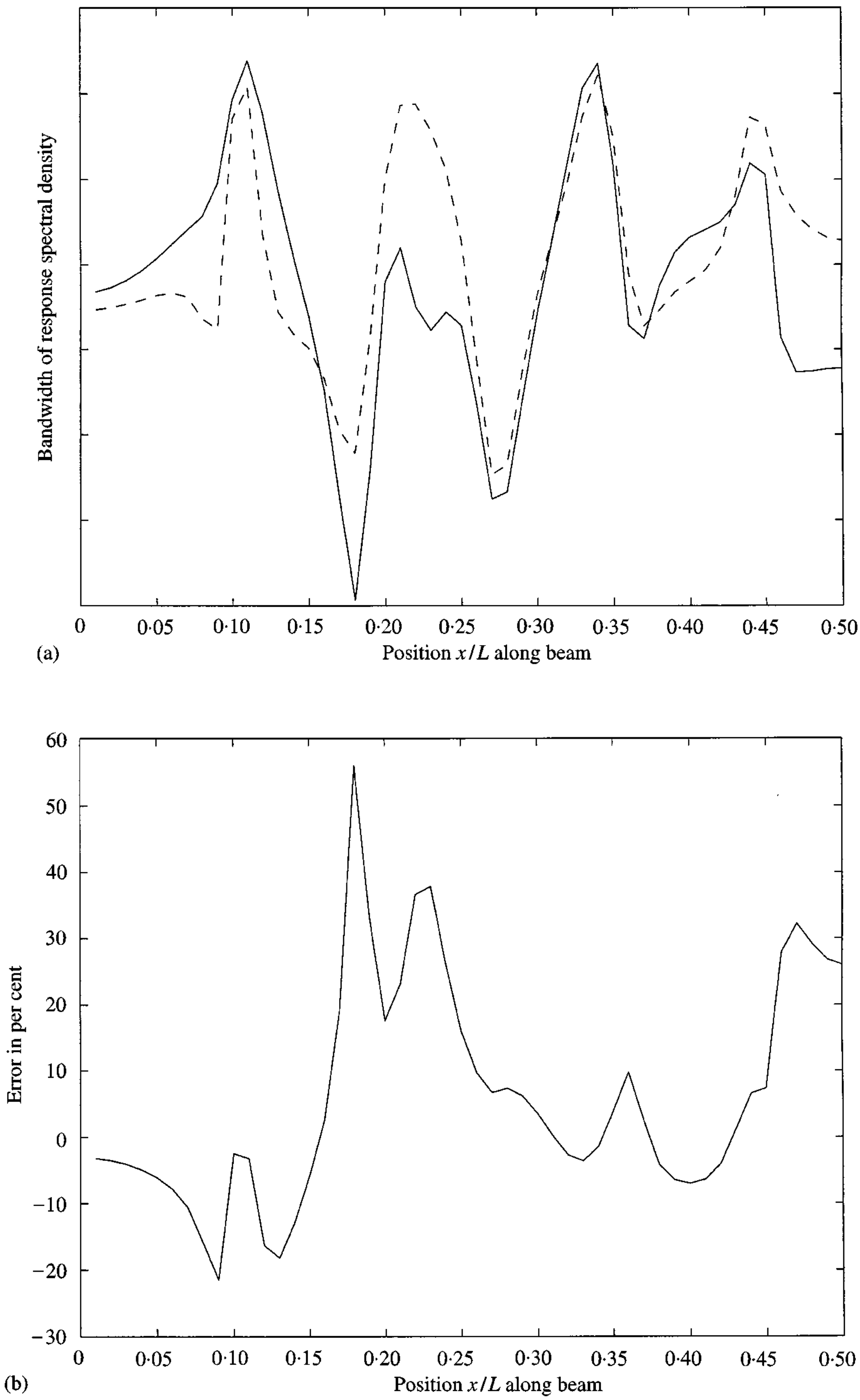



Figure 4. (a) Bandwidth of response spectral density of beam deflection $y(x, t)$ for $x/L < \frac{1}{2}$ (symmetry for $x/L > \frac{1}{2}$). The full line curve includes the modal cross-spectral terms as given in equation (24), and the dashed curve gives the approximate solution according to equation (25). (b) Error (in per cent) obtained when modal cross-spectral terms are neglected.

where $N_{max}$ is the number of maxima of the process during the time period $T$ and the bandwidth parameter $\varepsilon$ is defined as

$$\varepsilon = \sqrt{1 - \frac{m_2^2}{m_0 m_4}}. \tag{33b}$$

In the bandwidth definition (33b) the moments $m_i$ ($i = 0, 2, 4$) of the spectral density are

$$m_i = \int \omega^i S_{yy}(\omega)\,\mathrm{d}\omega. \tag{33c}$$

Using that $\sigma_y^2 = m_0$, $\sigma_{\dot{y}}^2 = m_2$ and $\sigma_{\ddot{y}}^2 = m_4$, one obtains $N_{mean} = \sigma_{\dot{y}} T/2\pi\sigma_y$. (The peak factor in equation (32) is sometimes seen with $N_{max}$ instead of $N_{mean}$ in the formula. It is then valid for narrow-band ($\varepsilon = 0$) processes only, as then $N_{max} = N_{mean}$, see equation (33a).)

The bandwidth $\varepsilon$ of the response process $y(x, t)$ is shown in Figure 4(a). It is seen that the bandwidth varies considerably along the beam (the bandwidth of the excitation process was $\varepsilon = 0{\cdot}49$). The error due to the exclusion of the modal cross-spectral densities is between $-20$ and $+55$ per cent, Figure 4(b). The peak factor (the factor multiplying the standard deviation $\sigma_y$ in equation (32)) does not, however, vary very much. Using $T = 50$ s, one obtains the peak factor as being close to 3·77, see Figure 5(a). The mean value of the maximum deflection of the response during the time period $T$ will then be close to $3{\cdot}77\sigma_y$, Figure 5(b), and the error due to the approximation is very close to the error of the standard deviation itself, i.e., between $-40$ and $+60$ per cent, see Figure 2(b).

## 3. DISCUSSION

By studying the vibration of a simply supported beam subjected to a stationary random loading it has been demonstrated that the modal cross-spectral densities may play an important role in a structural response even if the system is lightly damped and the eigenfrequencies are well separated. In the calculations made here, the random excitation covered the frequency range from 50 to 160 rad/s. This frequency range includes the five eigenfrequencies $\omega_{ej}$ where $j = 8, 9, 10, 11$ and 12 (of which only the two odd-numbered eigenmodes were excited by the random loading). Similar results should be obtained if the excitation is coloured noise with low intensity outside the given frequency range and high intensity within the interval.

If the main part of the excitation spectral density falls within the frequency range $\omega_{e7} = 49$ rad/s and $\omega_{e13} = 169$ rad/s, then it is easy to see from Figure 1 that there could be a significant difference between the two response spectral densities. Unfortunately, nothing can be said about the sign of the terms neglected. In Figure 1(a), it is seen that the approximate spectral density gives too low values in the frequency range 49–169 rad/s at $x = 0{\cdot}1L$ whereas it gives too large values at $x = 0{\cdot}45L$, Figure 1(b). At other positions along the beam the modal cross-spectral terms may cancel each other.

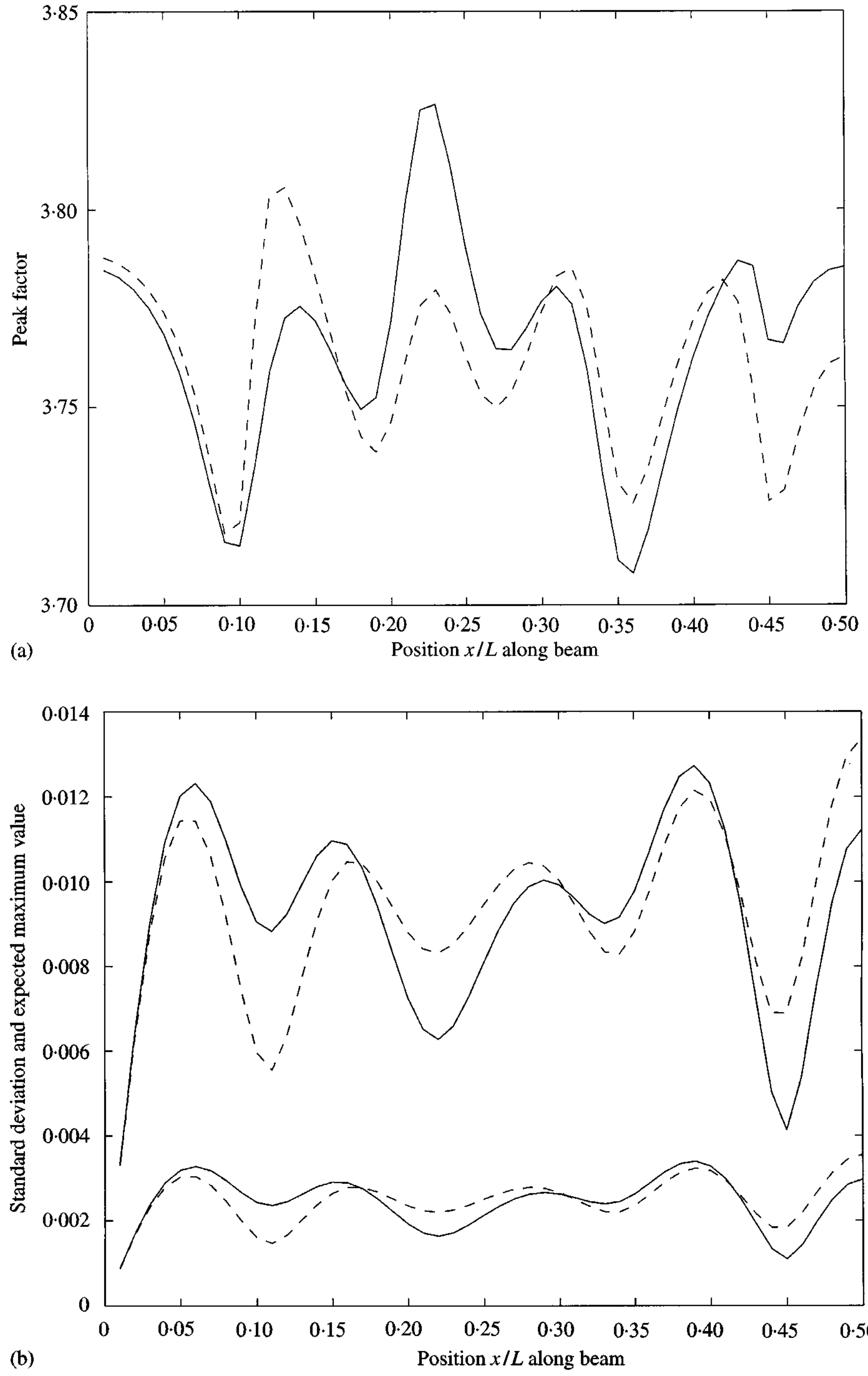



Figure 5. (a) Peak factor of beam deflection $y(x, t)$ for $x/L < \frac{1}{2}$ (symmetry for $x/L > \frac{1}{2}$). (b) Expected mean of maximum value of the beam deflection $y(x, t)$ during time periods $T = 50$ s. Also replotted in (b) is the standard deviation of the beam deflection $y(x, t)$ (from Figure 2). The error of the peak factor is small compared to the error of the standard deviation. The error of the expected maximum value is therefore similar to the error of the standard deviation, see Figure 2(b). The full line curves include the modal cross-spectral terms as given in equation (24), and the dashed curves give the approximate solution according to equation (25).

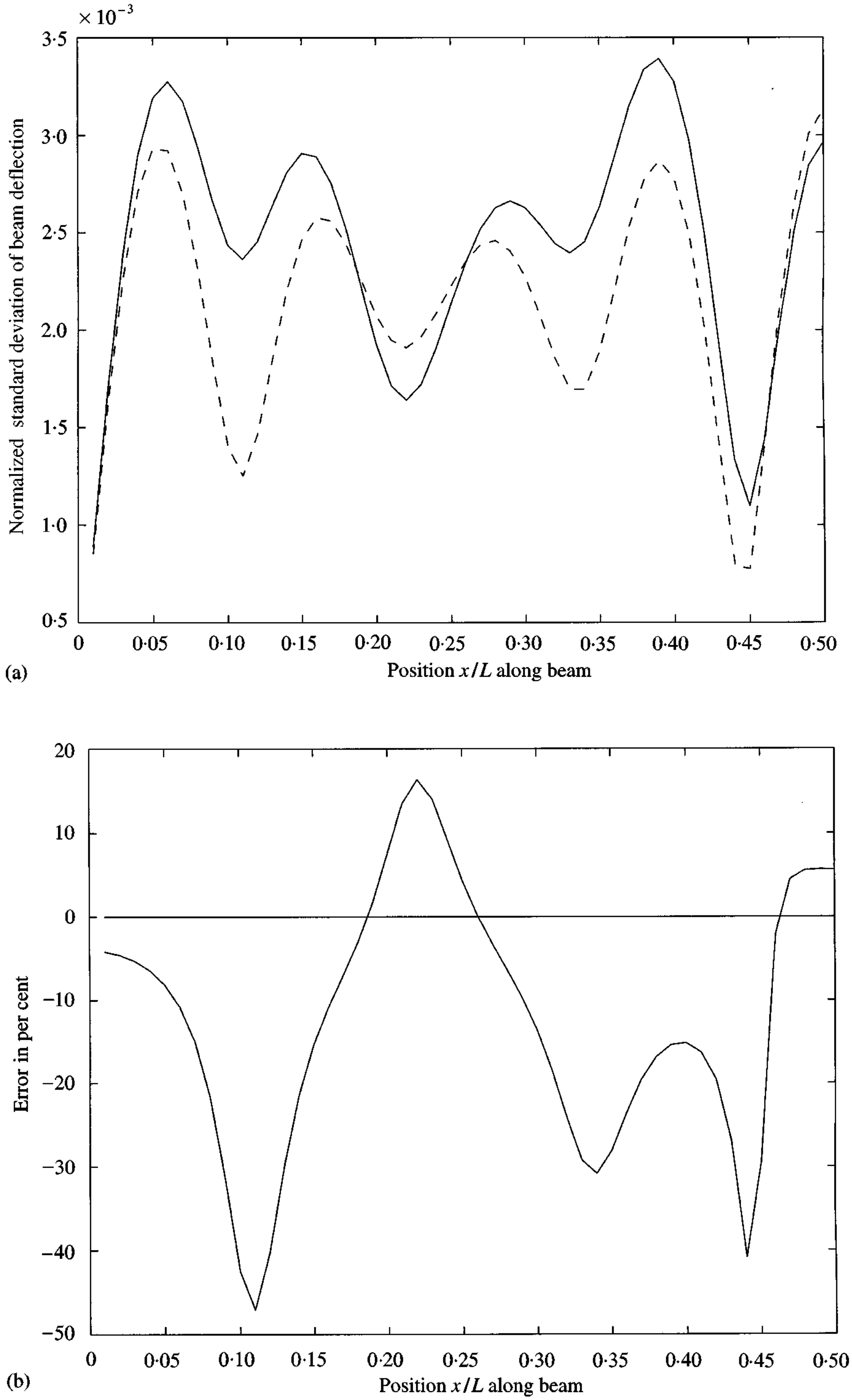



Figure 6. Standard deviation $\sigma_y$ of beam deflection $y(x, t)$. (a) equation (29) with summation over modes 1–59 (full line curve, same as full line curve in Figure 2) and equation (29) with summation over modes 7–13 only (dashed curve). (b) Percentage error due to the modal truncation.

The discrepancy between the approximate and the exact spectral density may be very large if the excitation spectral density is narrow-band with frequencies between two adjacent eigenfrequencies, see Figure 1.

In this study, the frequency interval of integration was selected to demonstrate that the influence of the modal cross-spectral terms may play an important role in the structural response. In other frequency ranges, the contributions from the cross-spectral terms may cancel each other. For example, excitation of the structure (and the corresponding integration) in the frequency range from $\omega_1 = 5$ to $\omega_2 = 200$ rad/s yields an error of the standard deviation $\sigma_y$ that is less than (or equal to) 6 per cent. If the first eigenmode is excited, that mode will dominate the response and the error due to the exclusion of the cross-spectral terms will be very small.

Another case when the error due to the neglect of the cross-spectral terms may be small is when the number of modes included in the calculation is small. In that case, however, the truncation error will probably be large. In the example studied here, taking only modes 7–13 into account (covering the excitation frequency range 50–160 rad/s) would give a truncation error of the standard deviation of the beam deflection as presented in Figure 6. In this case, the additional error made by omitting the cross-spectral terms would be very small; an extra error of less than 3 per cent was obtained when omitting the cross-spectral terms.

## 4. CONCLUSIONS

By a simple example, it has been demonstrated that the two conditions

(a) the system should be lightly damped and
(b) the eigenfrequencies should be well separated

are not sufficient to ensure that the modal cross-spectral densities may be neglected when calculating the response spectral density of a structure even if the random excitation of the structure is broadband so that several eigenmodes of the structure are excited.

In the calculations performed here, it has been demonstrated that the error made when neglecting the modal cross-spectral densities may be very large. Although the excitation frequency range covered several eigenfrequencies of the structure (a beam), an error of more than 60 per cent of the standard deviation of the beam deflection has been demonstrated. Factors that influence the cross-modal contributions are not only the damping and the eigenfrequency separation, but also the loading (its frequency content and its location on the structure), and the location where the response is calculated.

When rejecting modal cross-spectral densities in a calculation, it should be kept in mind that a large number of terms are omitted. Even if every term omitted is small, the sum of the terms need not be small. Also, at a particular point of the structure, where the response is calculated, the contributions from the direct modal spectral densities need not be large (one, or several, of the modes might have a node at that point). Therefore, there is no reason to recommend any rejection of the modal cross-spectral densities.

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## APPENDIX A: NOMENCLATURE

| | |
|---|---|
| $A_l$ | amplitude in mode $l$ |
| $c$ | damping (N s/m) |

| | |
|---|---|
| $E[\ ]$ | expectation |
| $EI$ | beam bending stiffness ($\mathrm{N\,m^2}$) |
| $f$ | distributed force (N/m) |
| $F$ | point force (N) |
| $\mathrm{i}$ | imaginary unit |
| $j$, $l$ | indices |
| $L$ | beam length (m) |
| $m$ | mass distribution (kg/m) |
| $m_i$ | $i$th moment of spectral density |
| $N_{\max}$, $N_{\mathrm{mean}}$ | number of maxima, number of mean value up-crossings |
| $\Re$ | real part of |
| $R(\tau)$ | auto-, cross-correlation |
| $s$ | space co-ordinate (m) |
| $S(\omega)$ | means square spectral density |
| $t$ | time co-ordinate (s) |
| $T$ | time period (s) |
| $x$ | space co-ordinate (m) |
| $y$ | beam deflection (m) |
| | |
| $\beta$ | damping parameter ($c/m$) |
| $\gamma$ | wave number (rad/m) (also Eulers constant $\gamma = 0{\cdot}57721 \ldots$ ) |
| $\delta$ | Dirac delta function |
| $\varepsilon$ | bandwidth ( $-$ ) |
| $\zeta$ | modal damping ( $-$ ) |
| $\lambda$ | wavelength ($\lambda = 2\pi/\gamma$) |
| $\xi$ | space separation (m) |
| $\sigma$ | standard deviation (m, m/s, $\mathrm{m/s^2}$) |
| $\tau$ | time separation (s) |
| $\omega$ | angular frequency (rad/s) |
| | |
| $(\dot{\ })$ | time differentiation |
| $(\ )^*$ | complex conjugate |
| | (some notations are defined where they appear) |